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LETTER TO THE EDITOR

Calculation of Lyapunov exponent using an equivalent stochastic system

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Abstract

It is conjectured that for every deterministic system there is a stochastic system that yields the same correlation functions. We exploit this to set up a scheme for the analytic computation of the Lyapunov exponent of a forced double-well oscillator. The resulting expression shows that the Lyapunov exponent has a maximum as a function of the well depth. Numerical evaluation of the Lyapunov exponent confirms this expectation.

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In an interesting work [1, 3] on possible connections between chaotic flows and statistical mechanics, it was argued some years ago that the Lyapunov exponent will obey a Kubo type formula for transport coefficients. This meant that the Lyapunov exponent could be written down as the integral over a two-time correlation function of some ‘current’. In arguing that this Kubo relation does give the correct Lyapunov exponent, the dynamics of the random variable had to be obtained and then the solution was used to (i) extract the Lyapunov exponent by the standard technique of Bennetin and Galgani [4] and (ii) extract the Lyapunov exponent from the Kubo formula [3]. The two techniques yielded the same exponent, showing the correctness of the Kubo formula. In this Letter we want to show that an ‘equivalent’ stochastic system [6][1], whose probability distribution is known, can be used to calculate the correlation function in the Kubo formula and yield a Lyapunov exponent which is very close to the actual one. This is obviously an efficient method since it obviates the necessity of solving for the original dynamics and should be able to shed light on features of the Lyapunov exponent which would not otherwise be apparent.

The Hamiltonian for a particle moving in a potential $V(x)$ can be written as

$$H = \frac{p^2}{2} + V(x). \tag{1}$$

[1] In a different context (the Kuramoto–Shivashinsky equation) a somewhat similar philosophy is applied in [5].

The equation of motion for the above particle is

$$\ddot{x} = -V'(x) \tag{2}$$

and is always integrable. If there is an additional time-dependent term in the Hamiltonian, then it is possible for the system to show chaotic behaviour if the potential $V(x)$ has certain characteristics.

The largest Lyapunov exponent for the dynamical system would be given by

$$\lambda = \lim_{\substack{t\to\infty \\ d(0)\to 0}} \frac{\log \|d(t)\|/\|d(0)\|}{t} \tag{3}$$

where $\|d(t)\|$ is defined as

$$\|d(t)\| = [\dot{\Delta}x^2 + \Delta x^2]^{\frac{1}{2}} \tag{4}$$

and would be positive when the system is chaotic. The classical Hamiltonian, which is capable of showing chaotic behaviour and is still a very popular model, can be written as

$$H = \frac{P^2}{2m} + V(x) + gx\cos\omega t. \tag{5}$$

Since Δx and $\dot{\Delta}x$ are the quantities that we need, it is worthwhile to write down the equation for Δx. It is clear that

$$\ddot{\Delta}x = V''(x)\Delta x. \tag{6}$$

Since the class of $V(x)$ which would show chaos is characterized by the existence of at least two equilibrium points, we write the above equation in a form which refers it to an equilibrium point x^*,

$$\ddot{\Delta}x + (V''(x^*) + [V''(x) - V''(x^*)])\Delta x = 0. \tag{7}$$

$V''(x^*)$ is a number, in fact a positive number when $V(x^*)$ is a minimum, and hence in the absence of $V''(x) - V''(x^*)$ we should have the equation of motion of an oscillator. However we are in the fully chaotic region for x and hence $V''(x) - V''(x^*)$ is actually a random variable which we write as $V''(x*)\zeta(t)$. Consequently Δx follows a stochastic equation of motion [2]

$$\frac{\mathrm{d}^2\Delta x}{\mathrm{d}\tau^2} + \omega^2(\tau)\Delta x = 0 \tag{8}$$

where $\tau = [V''(x^*)]^{\frac{1}{2}}t$ and $\omega^2 = 1 + \zeta(\tau)$. We can write this as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix}\Delta x \\ \dot{\Delta}x\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -\omega^2 & 0\end{pmatrix}\begin{pmatrix}\Delta x \\ \dot{\Delta}x\end{pmatrix}. \tag{9}$$

For the second moments, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix}\langle\Delta x^2\rangle \\ \langle\dot{\Delta}x^2\rangle \\ \langle\Delta x\rangle\langle\dot{\Delta}x\rangle\end{pmatrix} = [A_0 + \zeta(t)B]\begin{pmatrix}\langle\Delta x^2\rangle \\ \langle\dot{\Delta}x^2\rangle \\ \langle\Delta x\rangle\langle\dot{\Delta}x\rangle\end{pmatrix} \tag{10}$$

where

$$A_0 = \begin{pmatrix}0 & 0 & 2 \\ 0 & 0 & -2 \\ -1 & 1 & 0\end{pmatrix} \tag{11}$$

and

$$B = \begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & -2 \\ -1 & 0 & 0\end{pmatrix}. \tag{12}$$

The equation for the second moments can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix}\langle\Delta x^2\rangle\\ \langle\dot{\Delta} x^2\rangle\\ \langle\Delta x\rangle\langle\dot{\Delta} x\rangle\end{pmatrix}=\begin{pmatrix}0 & 0 & 2\\ C_3 & -C_2 & -2-2C\\ -1-C+C_1 & 1 & -C_2\end{pmatrix}\begin{pmatrix}\langle\Delta x^2\rangle\\ \langle\dot{\Delta} x^2\rangle\\ \langle\Delta x\rangle\langle\dot{\Delta} x\rangle\end{pmatrix} \tag{13}$$

where

$$C_1=\int_0^\infty \langle\langle\zeta(\tau)\zeta(\tau-\tau')\rangle\rangle \sin 2\tau'\,\mathrm{d}\tau' \tag{14}$$

$$C_2=\int_0^\infty \langle\langle\zeta(\tau)\zeta(\tau-\tau')\rangle\rangle(1-\cos 2\tau')\,\mathrm{d}\tau' \tag{15}$$

$$C_3=\int_0^\infty \langle\langle\zeta(\tau)\zeta(\tau-\tau')\rangle\rangle(1+\cos 2\tau')\,\mathrm{d}\tau' \tag{16}$$

$$C=\langle\zeta(t)\rangle \tag{17}$$

$$\langle\langle\zeta(\tau)\zeta(\tau-\tau')\rangle\rangle=\langle\zeta(\tau)\zeta(\tau-\tau')\rangle-\langle\zeta(\tau)\rangle\langle\zeta(\tau-\tau')\rangle. \tag{18}$$

In terms of the above constants the eigenvalue up to second order is

$$\lambda_0=\frac{(C_3-C_2)}{2}. \tag{19}$$

Next we evaluate $\zeta(t)$. We specialize to the potential $V(x)=ax^4-bx^2$.

The equilibrium fixed points are $x=0$, $x=\pm\sqrt{\frac{b}{2a}}$ of which $x=\pm\sqrt{\frac{b}{2a}}$ are stable:

$$\begin{aligned}\zeta(t)&=\left\{V''(x)-V''\left(\pm\sqrt{\frac{b}{2a}}\right)\right\}\Big/V''\left(\sqrt{\frac{b}{2a}}\right)\\ &=12a\left(x^2-\frac{b}{2a}\right)\Big/4b\\ &=\frac{3a}{b}\left(x^2-\frac{b}{2a}\right).\end{aligned} \tag{20}$$

At this point, we should like to evaluate the correlation functions shown in equations (14)–(17) by introducing an equivalent stochastic differential equation governing the dynamics of the variable x. What dictates the choice of the system? We should like to introduce the stochastic system in such a manner that the equilibrium probability distribution of that system is similar to the invariant probability distribution of the chaotic system described by equation (5). The distribution in this case corresponds to a dynamics which hops erratically between the two minima of $V(x)=ax^4-bx^2$. A stochastic system which has similar dynamics is the Langevin system

$$\dot{x}=-V'(x)+f \tag{21}$$

where f is a random noise, white and Gaussian and specified by the correlator

$$\langle f(t_1)f(t_2)\rangle=2\epsilon\delta(t_1-t_2). \tag{22}$$

and ϵ is the amplitude of the noise correlation. The equilibrium distribution is

$$P(x)\propto \mathrm{e}^{-V(x)/\epsilon} \tag{23}$$

and we can choose ϵ in such a manner that the transit time from one minimum to another matches the transit time in the chaotic systems. The transit time for the stochastic system is given by the Kramer formula

$$\begin{aligned}T&=\frac{2\pi}{\sqrt{|V''(x^*)||V''(0)|}}\mathrm{e}^{-\Delta V/\epsilon}\\ &=\frac{2\pi}{2\sqrt{2}b}\mathrm{e}^{b^2/4a\epsilon}.\end{aligned} \tag{24}$$

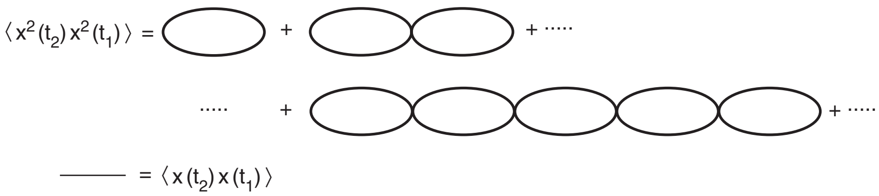

Figure 1. The diagrammatic representation of the correlation function $\langle x^2(t_1)x(t_2)\rangle$ in the spherical limit.

For $b = 10$ and $a = 0.5$ and the external forcing frequency $\omega = 6.07$ (which sets the timescale for transit in the dynamical system), we find $\frac{b^2}{4a\epsilon} \simeq 1.0$. To find the eigenvalue λ_0, we need to evaluate

$$\lambda_0 = \int \langle\langle \zeta(\tau')\zeta(\tau + \tau')\rangle\rangle \cos 2\tau \, \mathrm{d}\tau. \tag{25}$$

Noting that the integral of $\cos 2\tau$ is zero, we are, after straightforward algebra, left with the result

$$\lambda_0 = \frac{9a^2}{b^2} \int \langle\langle x^2(\tau')x^2(\tau + \tau')\rangle\rangle \cos 2\tau \, \mathrm{d}\tau. \tag{26}$$

The Lyapunov λ is related to λ_0 by the relation $\lambda = \frac{\lambda_0}{2}$.

The dynamics $\dot{x} = 2bx - 4ax^3 + f(t)$ does not allow for any perturbative solution at all. However, various nonperturbative techniques [9, 10] have been widely applied to this system and we have used this to find

$$\begin{aligned} C(12) &= \langle x(t_1)x(t_2)\rangle \\ &= \langle x^2\rangle_{st} \mathrm{e}^{-2b|t_1 - t_2|} \end{aligned} \tag{27}$$

where

$$\langle x^2\rangle_{st} \simeq \frac{b}{2a}(1 - a\epsilon/b^2). \tag{28}$$

We now need to relate $\langle x^2(t_1)x^2(t_2)\rangle$ to the correlation function $C(12)$. If the probability distribution which is used to calculate the correlation function is Gaussian, then the correlation function $\langle x^2(t_1)x^2(t_2)\rangle = 2\langle x(t_1)x(t_2)\rangle^2 = 2C_{12}^2$. However, this is an immense simplification. The probability distribution has strong non Gaussian behavior and hence the above result will be modified. The simplest situation in which a non-perturbative result can be obtained is the spherical limit. In this case one generalizes from the single component variable $x(t)$ to a N component vector $x_i(t)$, and for each i writes down the equation of motion

$$\dot{x_i} = 2bx_i - \frac{4a}{N}x^2 x_i + f_i \tag{29}$$

where $x^2 = \sum_{i=1}^{N} x_i^2$ and the noise f_i has no cross correlation, i.e.

$$\langle f_i(t) f_j(t')\rangle = 2\epsilon\delta_{ij}\delta(t - t').$$

The factor N in the denominator of the second term on the right-hand side of equation (29) ensures that the limit $N \to \infty$ is well defined. This is the spherical limit and in the limit all the correlation function can be related exactly to $C(12)$, which is exhibited in equation (27).

To relate $\langle x_i^2(t_1)x_i^2(t_2)\rangle$ (no sum on i) to $C(12)$, it is best to work in the frequency space. To the lowest order the Fourier transform $F(\omega)$ of $\langle x_i^2(t_1)x_i^2(t_2)\rangle$ is

$$F(\omega) = \frac{2\langle x^2\rangle_{st}^2/N}{-\mathrm{i}\omega + 4b}. \tag{30}$$

With the dynamics governed by equation (5) the only surviving graphs in the limit $N \to \infty$ are shown in figure 1. The result is

$$\begin{aligned} F(\omega) &= \left[\frac{2\langle x^2\rangle_{st}^2/N^2}{-\mathrm{i}\omega + 4b}\left[1 - 4a\frac{\langle x^2\rangle_{st}^2}{N^2}\right]\frac{1}{-\mathrm{i}\omega + b} + \left\{4a\frac{\langle x^2\rangle_{st}^2}{N^2}\right\}^2 \frac{1}{(-\mathrm{i}\omega + b)^2} + \cdots\right] \\ &= \frac{2\langle x^2\rangle_{st}^2/N^2}{-\mathrm{i}\omega + 4b}\frac{1}{1 + \frac{4a\langle x^2\rangle_{st}^2/N}{-\mathrm{i}\omega+4b}} = \frac{2\langle x^2\rangle_{st}^2/N^2}{-\mathrm{i}\omega + 4b + 4a\frac{\langle x^2\rangle_s^2 t}{N^2}} \\ &= \frac{2\langle x^2\rangle_{st}^2/N^2}{-\mathrm{i}\omega + 4b + 4a(\frac{b}{2a})^2} = \frac{2\langle x^2\rangle_{st}^2/N^2}{-\mathrm{i}\omega + 4b(1 + \frac{b}{4a})}. \end{aligned} \tag{31}$$

The final result shows that the screening effect increases the relaxation rate by the factor $1 + \frac{b}{4a}$. In real time, we have

$$\langle x^2(t_1)x^2(t_2)\rangle = 2\langle x^2\rangle_{st}^2 \exp\left(-4b\left(1 + \frac{b}{4a}\right)(t_2 - t_1)\right). \tag{32}$$

We are now in a position to evaluate the Lyapunov exponents. We find

$$\begin{aligned} \lambda &= \tfrac{1}{2}\lambda_0 \\ &= \frac{9}{2}\frac{a^2}{b^2}\int \langle x^2(t_1)x^2(t_2)\rangle \cos 2\tau \,\mathrm{d}\tau \\ &= 9\frac{a^2}{b^2}\langle x^2\rangle_{st}^2 \int \mathrm{e}^{-2\sqrt{b}(1+b/4a)\tau} \cos 2\tau \,\mathrm{d}\tau \\ &= 9\frac{a^2}{b^2}\langle x^2\rangle_{st}^2 \frac{\sqrt{b}(1 + b/4a)}{2[1 + b(1 + b/4a)^2]} \\ &= \frac{9}{8}\left(1 - \frac{a\epsilon}{b^2}\right)^2 \frac{\sqrt{b}(1 + b/4a)}{[1 + b(1 + b/4a)^2]} \end{aligned} \tag{33}$$

which is the central result of the paper.

We expect $a\epsilon/b^2$ to be small and from equation (24) estimate it to be about 0.1. In our method of setting up an equivalent random system, the strength of the sinusoidal term does not play a role so long as the strength is above the thresold for the onset of chaos. The determination of λ from the algorithm of Benettin *et al* is fairly independent of g. The driven system with $V = ax^4 - bx^2$ was extensively studied by Lin and Ballentine [12] and using their parameters $a = 0.5$ and $b = 10$ one obtains the chaotic behaviour of the system represented by equation (5) depending on the starting point. Equation (33) gives $\lambda = 0.048$. The value of λ found from the algorithm of Bennetin and Galgani [4] ranges from 0.0395 to 0.0449. The agreement is impressive. We find from equation (33) that $\lambda \propto \frac{1}{b^{3/2}}$ for large b. For very large b, the system is expected to be nonchaotic (the wells are very deep). For $b \to 0$, $\lambda \propto b^{\frac{1}{2}}$ and this implies that λ as a function of b should exibit a maximum. To test this we carried out a numerical calculation of λ based on the work of Bennetin *et al* for the actual system for various values of b. The results are shown in table 1 and the Lyapunov index does have a peak as a function of b. We thus establish that the equivalent stochastic system gives a good account of the correlation function of the deterministic system exhibiting chaos.

Table 1. The Lyapunov index λ for the chaotic trajectories of $\ddot{x} = 2bx - 4ax^3 + gx\cos\omega t$ as a function of b and g.

b	g	λ
5	3	0.010 6
6	3	0.003 6
6.5	3	0.256 0
7	3.5	0.003 6
7.5	3.5	0.029 0
8	4.5	0.000 1
8.5	5	0.002 0
9	6	0.013 0
10	6	0.007 0
10	9	0.041 3
10	10	0.039 5
10	10.5	0.041 8
10	11	0.043 4
10	11.5	0.044 9
10	12	0.040 3
13	12	0.047 0
13.5	12	0.045 8
14	12	0.047 1
15	9	−0.000 06
15	12	0.014 0
16	12	0.001 7
17	12	−0.001 6

Recently the system represented by equation (5) has been studied in connection with decoherence and entropy production in chaotic systems [8] and stochastic resonance in a bistable system subject to multiplicative and additive noise [7, 13]. Further investigation of the latter from a quantum mechanical point of view is currently in progress.

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